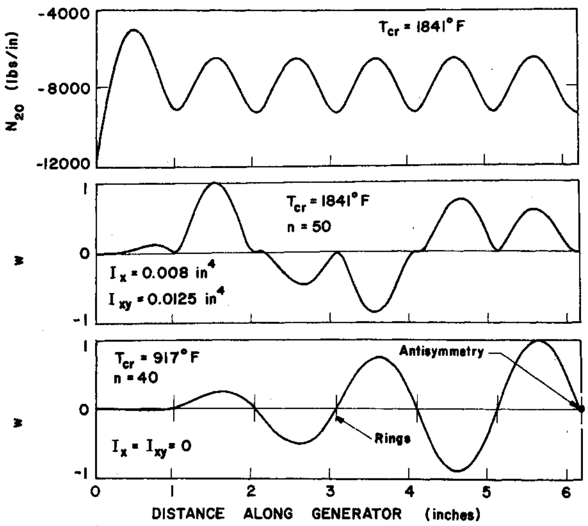

Fig. 5 Prebuckling hoop stress resultant N_{20}, buckling temperatures T_{cr}, circumferential waves n, and modes for case 2 (ring spacing = 1.025 in., see Fig. 2).

(17e) of Ref. 5, this moment of inertia is contained in a term of the ring strain energy which is multiplied by the fourth power of the circumferential wave number n. In these cases $35 \leq n \leq 75$, so that this term represents a significant contribution to the total strain energy of the system.

Note from Figs. 4 and 5 that if I_z is neglected, buckling is antisymmetric about the rings; and if I_z is accounted for, buckling is symmetric about the rings. The second term in Eq. (17c) of Ref. 5 causes this difference in predicted behavior. Additional of the effect of I_z essentially leads to a clamped condition at the ring stations. It is emphasized that this phenomenon will be less apparent if buckling is associated with a smaller number of circumferential waves.

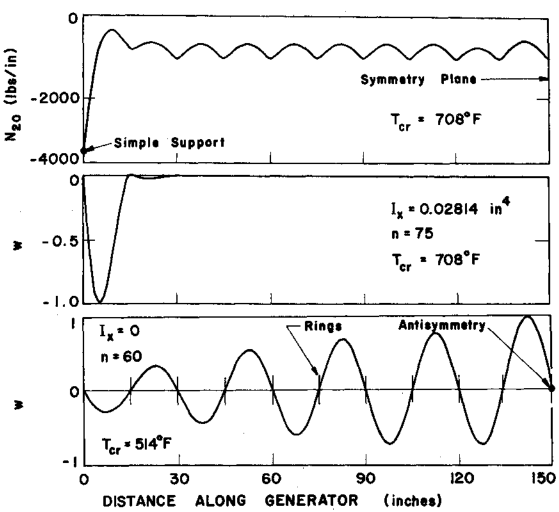

Fig. 6 Prebuckling hoop stress resultant N_{20}, buckling temperatures T_{cr}, circumferential waves n, and modes for case 3 (see Fig. 3).

References

[1] Anderson, M. S., "Thermal Buckling of Cylinders," *Collected Papers on Instability of Shell Structures*, TN D-1510, Dec. 1962, NASA, pp. 255–265.

[2] Johns, D. J., "Local Circumferential Buckling of Thin Circular Cylindrical Shells," *Collected Papers on Instability of Shell Structures*, TN D-1510, Dec. 1962, NASA, pp. 266–276.

[3] Chang, L. K. and Card, M. F., "Thermal Buckling in Stiffened Cylindrical Shells," *Proceedings of the AIAA/ASME 11th Structures, Structural Dynamics, and Materials Conference*, April 1970, pp. 260–272.

[4] Block, D. L., "Influence of Prebuckling Deformations and Discrete Ring Stiffeners on the Buckling of Eccentrically Stiffened Orthotropic Cylinders," TN D-4283, 1967, NASA.

[5] Bushnell, D., "Analysis of Ring-Stiffened Shells of Revolution Under Combined Thermal and Mechanical Loading," *Proceedings of the AIAA/ASME 11th Structures, Structural Dynamics, and Materials Conference*, April 1970, pp. 196–210; also *AIAA Journal*, Vol. 9, No. 3, March 1971, pp. 401–410.

Empirical Bayes State Estimation in Discrete Time Linear Systems

H. F. Martz Jr.*
Texas Tech University, Lubbock, Texas

AND

S. J. Kamat†
New Mexico State University, Las Cruces, N. Mex.

Introduction

EMPIRICAL Bayes decision theory is used here to develop a filter set for estimating the state of a linear dynamic system which does not require any assumption about the form of the underlying state disturbance error distribution or any of its moments. The filter is applicable regardless of the form of the unknown state error distribution and has been shown to provide rather stable performance characteristics for a variety of different shaped state error distributions. Some distributional assumptions on the initial state vector are also relaxed. The mean of this initial state vector is required for starting the estimation process, but knowledge of the covariance matrix and the form of the distribution is unnecessary. Thus fewer total assumptions are required to implement the filter. Finally, the filter developed here may be combined with several existing procedures, for example as in Refs. 1 or 2, for the case where the observation error covariance matrices are unknown and are to be estimated.

Statement of the Problem

Consider the following linear discrete dynamic system

$$x_n = \phi_{n,n-1}x_{n-1} + u_{n-1} \tag{1}$$

with the linear set of observations on this system

$$y_n = H_n x_n + v_n \tag{2}$$

where n is a time index, x_n is a $r \times 1$ system state vector, y_n is a $p \times 1$ observation vector, $\phi_{n,n-1}$ is a $r \times r$ state transition matrix, and H_n is a $p \times r$ matrix relating x_n to y_n. In addition, it is assumed that 1) v_n dist. $N_p(0,R_n)$ (p-normal with mean vector 0 and covariance matrix R_n); 2) $\mathrm{Cov}(v_n,v_m) = \delta_{nm}R_n$; 3) u_{n-1} has an unknown and unspecified distribution which remains stationary over time; 4) u_n is independent of u_m for all $n \neq m$; 5) u_n is independent of v_m for all n and m; 6) $E(x_0) = c$; and 7) $\phi_{n,n-1},H_n,R_n,c$ are known. With these assumptions, the problem is to estimate x_n from the observations $y_1,y_2, \ldots, y_n$.

Received November 19, 1970; revision received April 9, 1971. This research was supported by NASA Grants NGR 44-001-031 and NGR 44-011-040.

* Associate Professor of Industrial Engineering and Statistics.

† Assistant Professor of Industrial Engineering.

Empirical Bayes State Estimation

A complete description of empirical Bayes estimation is provided in Refs. 3–6 and a brief summary is provided in Ref. 2. An empirical Bayes estimator for the mean θ_n of an r-normal distribution with known covariance matrix S_n is given by[7]

$$E_n(\theta|z_n) = \frac{P_n^{-1}\sum_{i=1}^{n} a_{in}\exp[\frac{1}{2}a_{in}^T P_n^{-1}a_{in} - \frac{1}{2}\theta_i^T\Gamma_n^{-1}\theta_i]}{\sum_{i=1}^{n}\exp[\frac{1}{2}a_{in}^T P_n^{-1}a_{in} - \frac{1}{2}\theta_i^T\Gamma_n^{-1}\theta_i]} \tag{3}$$

where a prior density approximation $g_n(\theta)$ has been used for the unknown prior density $g(\theta)$ and where

$$\Gamma_n = \mathrm{diag}[\gamma_1^2(n), \ldots, \gamma_r^2(n)];$$
$$\gamma_j(n) = n^{-1/25}\{\mathrm{Var}(\theta_j)\}^{1/2} \tag{4}$$

$$\mathrm{Var}(\theta_j) = \sum_{i=1}^{n}\frac{(\theta_{ij}-\bar{\theta}_j)^2}{n},\ \bar{\theta}_j = \sum_{i=1}^{n}\frac{\theta_{ij}}{n} \tag{5}$$

$$P_n = S_n^{-1} + \Gamma_n^{-1} \tag{6}$$

$$a_{in} = S_n^{-1}z_n + \Gamma_n^{-1}\theta_i \tag{7}$$

Here θ_i is the realized value of θ at the ith estimation stage and θ_{ij} is the jth component of θ_i.

Consider Eq. (1) in the form

$$u_{n-1} = x_n - \phi_{n,n-1}x_{n-1} \tag{8}$$

Let

$$q_n = y_n - H_n\phi_{n,n-1}x_{n-1} \tag{9}$$

Using Eq. (2), q_n can be rewritten as

$$q_n = H_n u_{n-1} + v_n \tag{10}$$

Define

$$r_n = (H_n^TH_n)^{-1}H_n^Tq_n \tag{11}$$

or

$$r_n = u_{n-1} + (H_n^TH_n)^{-1}H_n^Tv_n \tag{12}$$

where the transformation in Eq. (11) requires the matrix $(H_n^TH_n)$ to be of full rank. If $(H_n^TH_n)$ is not of full rank then the use of a suitable generalized inverse should be considered.[8] From assumption 1 following Eq. (2), it follows that, given u_{n-1},r_n will be conditionally normally distributed with mean vector u_{n-1} and covariance matrix

$$S_n = (H_n^TH_n)^{-1}H_n^TR_nH_n(H_n^TH_n)^{-1} \tag{13}$$

The empirical Bayes estimator in Eq. (3) can be used to estimate the state disturbance error u_{n-1}, the conditional mean of r_n, provided $x_0,x_1, \ldots, x_{n-1}$ are known. To use the empirical Bayes estimator at time t_n for estimating u_{n-1}, suitable estimates of $x_0,x_1, \ldots, x_{n-1}$ are required. Since the mean of x_0 is assumed to be c,$\hat{x}_0 = c$ is a suitable estimate to start the empirical Bayes filter, as no observations are available at time t_0. The estimates $\hat{x}_i$ obtained at times $t_1,t_2, \ldots, t_{n-1}$ by means of the empirical Bayes filter can then be used in lieu of x_i in forming the estimate of u_{n-1}.

Let

$$\bar{x}_n = \phi_{n,n-1}\hat{x}_{n-1} \tag{14}$$

We thus have

$$\hat{u}_{n-1} = \frac{B_n^{-1}\sum_{i=1}^{n} d_{in}\exp[\frac{1}{2}d_{in}^TB_n^{-1}d_{in} - \frac{1}{2}\hat{r}_i^T\Gamma_n^{-1}\hat{r}_i]}{\sum_{i=1}^{n}\exp[\frac{1}{2}d_{in}^TB_n^{-1}d_{in} - \frac{1}{2}\hat{r}_i^T\Gamma_n^{-1}\hat{r}_i]} \tag{15}$$

where

$$B_n = S_n^{-1} + \Gamma_n^{-1} \tag{16}$$

$$\hat{r}_i = (H_i^TH_i)^{-1}H_i^T(y_i - H_i\bar{x}_i) \tag{17}$$

$$d_{in} = S_n^{-1}\hat{r}_n + \Gamma_n^{-1}\hat{r}_i \tag{18}$$

and where S_n is given by Eq. (13). In addition, Γ_n is given by Eq. (4) in conjunction with Eq. (5) where θ_{ij} now becomes the jth component of the vector $\hat{r}_i$. After u_{n-1} has been obtained the estimate of x_n becomes

$$\begin{aligned}\hat{x}_n &= \phi_{n,n-1}\hat{x}_{n-1} + \hat{u}_{n-1}\\ &= \bar{x}_n + \hat{u}_{n-1}\end{aligned} \tag{19}$$

Improved Prior Density Approximation

By comparing Eqs. (3) and (15), we observe that $\hat{r}_i$ has been substituted for θ_i which is now represented by u_{i-1}. It is desirable that the prior density approximation $g_n(u)$ be as close as possible to the unknown density $g(u)$. If $\hat{r}_i$, from which $g_n(u)$ is formed, is such that

$$E(\hat{r}_i) = E(u_{i-1}) \tag{20}$$

and

$$\mathrm{Cov}(\hat{r}_i) = \mathrm{Cov}(u_{i-1}) \tag{21}$$

then the distributions of $\hat{r}_i$ and u_{i-1} will have a second order match and an improved prior density approximation should result. Unfortunately, $\hat{r}_i$ does not possess the second property above. At the nth estimation stage define

$$\bar{r}_n = \sum_{i=1}^{n}\hat{r}_i/n \tag{22}$$

$$C_n = \left\{\sum_{i=1}^{n}(\hat{r}_i - \bar{r}_n)(\hat{r}_i - \bar{r}_n)^T\right\}\Big/(n-1) - S_n \tag{23}$$

Thus $\bar{r}_n$ and C_n are estimates of the mean and covariance matrix of u_{n-1}, respectively, at stage n. Suppose we define A to be an $r \times r$ diagonal matrix with jth diagonal element

$$A_{jj} = \{[C_n]_{jj}/[C_n + S_n]_{jj}\}^{1/2} \tag{24}$$

where $[\ ]_{jj}$ refers to the jth diagonal element of the indicated matrix. Also define

$$b = (I - A)\bar{r}_n \tag{25}$$

By forming

$$\tilde{r}_i = A\hat{r}_i + b, i = 1, \ldots, n \tag{26}$$

we now have that the expected value of $\tilde{r}_i$ is approximately equal to the expected value of u_{i-1} and that the variance of each component of $\tilde{r}_i$ is approximately equal to the variance of the corresponding component of u_{i-1}. For small n (less than five) sampling error may cause some diagonal element of C_n to be negative which causes the corresponding diagonal element of A to be imaginary. If this should occur, the particular element of A is then set equal to one.

With this improved prior density approximation, the filter equations for estimating x_n based on $y_1, \ldots, y_n$ become

$$\hat{u}_{n-1} = \frac{B_n^{-1}\sum_{i=1}^{n} g_{in}\exp[\frac{1}{2}g_{in}^TB_n^{-1}g_{in} - \frac{1}{2}\tilde{r}_i^T\Gamma_n^{-1}\tilde{r}_i]}{\sum_{i=1}^{n}\exp[\frac{1}{2}g_{in}^TB_n^{-1}g_{in} - \frac{1}{2}\tilde{r}_i^T\Gamma_n^{-1}\tilde{r}_i]} \tag{27}$$

where

$$g_{in} = S_n^{-1}\hat{r}_n + \Gamma_n^{-1}\tilde{r}_i \tag{28}$$

and $\tilde{r}_i$ is given in Eq. (26) in conjunction with equations

(22–25). All other terms in this equation are the same as in Eq. (15) and $\hat{x}_n$ is given by Eq. (19).

Performance Characteristics

The performance characteristics of the empirical Bayes filter were investigated by Monte Carlo simulation for the case $p = r = 6$. The ratio of the trace of the averaged squared error matrix to the trace of the matrix S_n was used as a scalar measure of performance. Thus a decreasing performance ratio is associated with an increase in the performance of the filter.

In Ref. 5 it was found that the performance of an empirical Bayes estimator for estimating the mean of a multivariate normal distribution depended upon a summary quantity Z_j. If Q is the covariance matrix of u_n, the summary quantity here becomes

$$Z_j = [S_n - (Q^{-1} + S_n^{-1})^{-1}]_{jj}/[(Q^{-1} + S_n^{-1})^{-1}]_{jj} \quad (29)$$

An average summary quantity Z was defined as

$$Z = \sum_{j=1}^{r} Z_j/r \quad (30)$$

For a fixed n, it was found that the performance ratio just defined decreased as Z increased. Also, for a given situation, the performance ratio was observed to decrease monotonically with n until around $n = 25$ after which the performance ratio was essentially flat. These results agree with those of Ref. 5. It was also observed that the performance of this filter was fairly stable when components of u were generated from different shaped distributions except for a U-shaped distribution for which it was considerably better. This discrepancy is explained in Ref. 5.

Conclusions

An empirical Bayes filter has been developed for estimating the state of a discrete time linear system with linear observations. Some distributional assumptions on the state disturbance error as well as some distributional assumptions on the initial state vector have been relaxed. Knowledge of the forms of these distributions is not required for use of the filter. It is required that the state disturbance error be uncorrelated over time and be independent of the observation error and that the state error distribution remain stationary over time.

The performance of the filter has been examined by means of Monte Carlo simulation. It has been found that the performance of the filter does not depend significantly on the form of the state disturbance error distribution used in the simulation.

It has also been observed that the performance of the empirical Bayes filter depends on the relative magnitude of the observation error and state disturbance error covariance matrices, all other parameters being held fixed. An index has been presented which summarizes this relative magnitude, and the performance of the filter has been found to increase as this index increases.

References

[1] Weiss, I. M., "A Survey of Discrete Kalman-Bucy Filtering with Unknown Noise Covariances," AIAA Paper 70-955, Santa Barbara, Calif., 1970.

[2] Martz, H. F. and Born, G. H., "Empirical Bayes Estimation of Observation Error Variances in Linear Systems," AIAA Paper 70-1058, Santa Barbara, Calif., 1970.

[3] Robbins, H., "An Empirical Bayes Approach to Statistics," *Proceedings of the Third Berkeley Symposium on Probability and Statistics*, Vol. 1, 1955, pp. 157–163.

[4] Atchison, T. A. and Martz, H. F., eds., *Proceedings of the Symposium on Empirical Bayes Estimation and Computing in Statistics, Mathematics Series No. 6*, Texas Tech Press, Lubbock, Texas, 1970.

[5] Martz, H. F. and Krutchkoff, R. G., "Empirical Bayes Estimators in a Multiple Linear Regression Model," *Biometrika*, Vol. 56, No. 2, 1969, pp. 367–374.

[6] Wasan, M. T., *Parametric Estimation*, McGraw-Hill, New York, 1970.

[7] Kamat, S. J., *Application of Empirical Bayes Decision Procedures To Discrete Time Linear Filtering*, Ph.D. dissertation, 1970, Texas Tech Univ., Lubbock, Texas.

[8] Deutch, R., *Estimation Theory*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

Separation of a Supersonic Accelerated Flow over Notches

A. F. CHARWAT*
University of California, Los Angeles, Calif.

SEPARATED cavities in notches and grooves in the boundary of a uniform supersonic flow have been described extensively; the present Note discusses the effect of a negative pressure gradient (i.e., freestream acceleration) on the flow over such notches. Such situations occur in nozzle flows, near the nose of blunt hypersonic vehicles, etc.

Experiments were conducted in a wind tunnel fitted with nozzle blocks which generated a linear negative pressure gradient over the test section. Reference 1 gives detailed pressure distributions over the floor of the cavity and impact-pressure profiles through the shear layer. The equipment is described in Ref. 1 (also Ref. 2, which reports on similar studies of down-stream-facing steps).

Rectangular notches 0.5 and 1 in. deep with variable lengths were tested. The Mach number immediately ahead of separation was 1.81. The pressure gradient (and therefore the Mach number gradient) in the undisturbed stream over the region of the notch is characterized by a scale length

$$\lambda = -[(P_t/P_0)d(P/P_0)/dx]^{-1} \quad (1)$$

where P_0 is the static pressure immediately upstream of separation. The values of λ^{-1} used in these experiments was ∞ (uniform flow), 9.17 and 5.77 (in.). The upstream boundary layer was fully turbulent and the ratio of its thickness relative to the notch depth varied between 0.2 and 0.4.

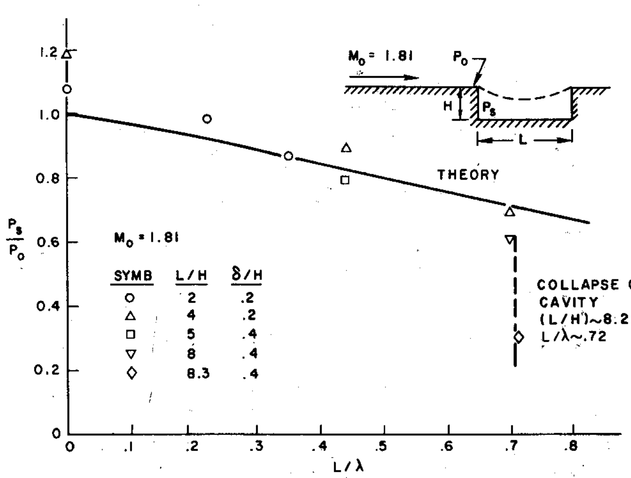

Fig. 1 Variation of the cavity pressure ratio in accelerated flow.

Received January 11, 1971; revision received April 23, 1971.

* Professor, School of Engineering and Applied Science.